AN ALGEBRAIC CHARACTERIZATION OF A DEHN TWIST FOR NONORIENTABLE SURFACES

FERIHE ATALAN

ABSTRACT. Let N_g^k be a nonorientable surface of genus $g \geq 5$ with k-punctures. In this note, we will give an algebraic characterization of a Dehn twist about a simple closed curve on N_g^k. Along the way, we will fill some little gaps in the proofs of some theorems in [1] and [4] giving algebraic characterizations of Dehn twists about separating simple closed curves. Indeed, our results will give an algebraic characterization for the topological type of Dehn twists about separating simple closed curves.

1. INTRODUCTION

In this note, $N_{g,r}^k$ will denote the nonorientable surface of genus g with r boundary components and k punctures (or distinguished points). The mapping class group of $N_{g,r}^k$, the group of isotopy classes of all diffeomorphisms of $N_{g,r}^k$, where diffeomorphisms and isotopies fix each point on the boundary is denoted by $\mathrm{Mod}(N_{g,r}^k)$. If we restrict ourselves to the diffeomorphisms and isotopies to those which do not permute the punctures then we obtain the pure mapping class group $\mathrm{PMod}(N_{g,r}^k)$. The subgroup $\mathrm{PMod}^+(N_{g,r}^k)$ of the pure mapping class group consists of the pure mapping classes that preserve the local orientation around each puncture. Also the twist subgroup of $\mathrm{Mod}(N_{g,r}^k)$, generated by Dehn twists about two-sided simple closed curves is denoted by $\mathcal{T}$.

An algebraic characterization of a Dehn twist plays important role in the computation of the outer automorphism group of the mapping class group of a surface (orientable or nonorientable, see [1], [2] and [4]). Moreover, it is one of the main tools in the proof of the fact that any injective endomorphism of the mapping class group of an orientable surface must be an isomorphism proved by Ivanov and McCarthy ([6]). We note that it is also used in the proof of the fact that any isomorphism between two finite index subgroups of the extended mapping class group of an orientable surface is the restriction of an inner automorphism of this group ([5], [8]). Using an

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algebraic characterization of Dehn twists, we show that any automorphism of the mapping class group of a surface takes Dehn twists to Dehn twists.

For orientable surfaces, N. V. Ivanov gave an algebraic characterization of Dehn twists in [4]. In December 2012, E. Irmak reported that the proofs of Ivanov's theorem on algebraic characterization of Dehn twists about separating simple closed curves have some gaps (see the counter example in Section 2).

For closed nonorientable surfaces, we gave an algebraic characterization of Dehn twists in [1], closely following Ivanov's work [4]. Therefore, the algebraic characterization about separating curves in [1] has also gaps. In [2], using different techniques, we gave an algebraic characterization for Dehn twists about nonseparating simple closed curves with nonorientable complements. However, an algebraic characterization of the Dehn twists about other two-sided simple closed curves is still missing. In this paper, we will try to extend this result to Dehn twists about arbitrary simple closed curves. Indeed, we will not only give an algebraic characterization of a Dehn twist about a simple closed curve on N_g^k but also an algebraic characterization of the topological type of the curve the Dehn twist is about. In particular, this paper aims to fill the gaps, mentioned above, in both [1] and [4].

The organization of the paper is as follows: In Section 2, firstly, we will give some definitions and remanding, secondly, we will state and prove an algebraic characterization for a power of a Dehn twist about a simple closed curve. In Section 3, after proving some preliminary results we will first characterize the Dehn twists about characteristic curves on a nonorientable surface of even genus. Then Lemma 3.8 will lead to an algebraic characterization of Dehn twists about separating curves (Theorem 4.1). Moreover, this algebraic characterization will encode the topological type of the separating simple closed curve the Dehn twist which is about. In Section 4, we will consider nonorientable surfaces, where as in Section 5 we will state analogous results for orientable surfaces. The final section contains some immediate consequences of the these results.

2. Preliminaries

Let S denote the surface N_g^k and let a be a simple closed curve on S. If a regular neighborhood of a is an annulus or a Möbius strip, then we call a a two-sided or a one-sided simple closed curve, respectively. The curve a will be called trivial, if it bounds a disc with at most one puncture or a Möbius band on S (or if it is isotopic to a boundary component). Otherwise, it is called nontrivial.

We will denote by S^a the result of cutting of S along the simple closed curve a. The simple closed curve a is called nonseparating if S^a is connected. Otherwise, it is called separating.

Let H be a group. If $G \leq H$ is a subgroup and $h \in H$ is an element of H, then the center of H, the centralizer of G in H and the centralizer of h in G will be denoted by $C(H)$, $C_H(G)$ and $C_G(h)$, respectively.

Let $P : \Sigma^{2k}_{g-1} \to S$ be the orientation double covering of S and $\tau : \Sigma^{2k}_{g-1} \to \Sigma^{2k}_{g-1}$ the Deck transformation, which is an involution. It is well known that any diffeomorphism $f : S \to S$ has exactly two lifts to the orientation double covering and exactly one of them is orientation preserving. Moreover, we can regard $\mathrm{Mod}(S)$ as the subgroup $\mathrm{Mod}(\Sigma^{2k}_{g-1})^{\tau}$, the subgroup of mapping classes which are invariant under the action of the deck transformation (see also [1]).

Let $\Gamma(m)$, where $m \in \mathbb{Z}$, $m > 1$, be the kernel of the natural homomorphism

$$\mathrm{Mod}(\Sigma^{2k}_{g-1}) \to \mathrm{Aut}\,(H_1(\Sigma^{2k}_{g-1}, \mathbb{Z}/m\mathbb{Z})).$$

Then $\Gamma(m)$ is a subgroup of finite index in $\mathrm{Mod}(\Sigma^{2k}_{g-1})$. Let $\Gamma'(m) = \Gamma(m) \cap \mathrm{Mod}(S)$, regarding $\mathrm{Mod}(S)$ as a subgroup of $\mathrm{Mod}(\Sigma^{2k}_{g-1})$ as described above.

It is well known that since $\Gamma(m)$ consists of pure elements only provided that $m \geq 3$, ([4]) and so does $\Gamma'(m)$. In this paper, we will consider the group $\Gamma'(m)$ only for $m \geq 3$. Moreover, Γ' will always denote a subgroup of finite index in $\Gamma'(m)$.

Suppose that $f \in \Gamma'(m)$ preserves a generic family of disjoint simple closed curves $\mathcal{C}$. Then we can assume that f is identity on all points of $\mathcal{C}$. In this case, f does not interchange components of $S^{\mathcal{C}}$ and it induces a diffeomorphism on every component of $S^{\mathcal{C}}$, which is either isotopic to a pseudo-Anosov one, or to the identity. In this case, $\mathcal{C}$ is called a reduction system for f (and, for its mapping class). Also, if reduction system for f is minimal, then $\mathcal{C}$ is called a minimal reduction system for f.

The following lemma was proved in [1] mainly following Ivanov's work.

Lemma 2.1. *The center of every subgroup of finite index of* $\Gamma'(m)$ *is trivial.*

Let $\mathrm{M}(S)$ be any of three groups $\mathrm{PMod}^+(S)$, $\mathrm{PMod}(S)$ and $\mathrm{Mod}(S)$.

Lemma 2.2. *Let* $S = N^k_g$ *be a connected nonorientable surface of genus* $g \geq 5$ *with* k *punctures. Let* f *be an element in* Γ'*. Then* f *is a power of a Dehn twist about a simple closed curve if and only if the following conditions are satisfied:*

(1) $C(C_{\Gamma'}(f)) \cong \mathbb{Z}$,
(2) $C(C_{\Gamma'}(f)) \neq C_{\Gamma'}(f)$.

Proof. Assuming that the above conditions are satisfied, we need to show that f is a power of a Dehn twist about a simple closed curve.

By the second condition, f cannot be pseudo-Anosov. Indeed, if f is pseudo-Anosov then the centralizer in Γ' of a pseudo-Anosov element is isomorphic to $\mathbb{Z}$. (We note that the centralizer in Γ of a pseudo-Anosov element is isomorphic to $\mathbb{Z}$ in [4].) Therefore, $C(C_{\Gamma'}(f)) = C_{\Gamma'}(f)$, and we

would get a contradiction. Now using the assumptions that $C(C_{\Gamma'}(f)) \cong \mathbb{Z}$ and f is a pure element, it is easy to see that f is a power of a Dehn twist about a simple closed curve.

For the other direction of the lemma, assume that f is a power of a Dehn twist about a simple closed curve c. To show the first condition we will use Ivanov's ideas. We see that $C_{\Gamma'}(f^n)$ is equal to the set of elements of Γ' with minimal reduction system containing c, up to isotopy. So, there is a natural map from $C_{\Gamma'}(f^n)$ to $\mathrm{Mod}(S^c)$. The kernel of this map consists of powers of the twist f. Also, the image of this map is of finite index subgroup containing in the pure subgroup $\Gamma'(m)$ of $\mathrm{Mod}(S^c)$. Hence, the center of this image is trivial by Lemma 2.1. Hence, $C(C_{\Gamma'}(f^n))$ consists of powers of f and so, it is isomorphic to $\mathbb{Z}$.

Now, let us verify the second condition. If f is a power of a Dehn twist about a separating simple closed curve c, then the curve c separates S in two surfaces with holes. One of the components is either a nonorientable surface of genus at least three or an orientable surface of genus at least one. Let N_1 denote this component. Clearly, N_1 contains two-sided simple closed curves d and e so that d and e intersect transversally one point. Obviously, t_d^n, t_e^n are in $C_{\mathrm{M}(S)}(f)$ for any integer n. Hence, if t_d^n, t_e^n are in Γ', t_d^n, t_e^n are in $C_{\Gamma'}(f)$. On the other hand, if $n > 0$ then the Dehn twists t_d^n and t_e^n don't commute, and hence t_d^n, t_e^n are not in $C(C_{\Gamma'}(f))$. So, we obtain second condition above.

Similar arguments work for nonseparating simple closed curves with orientable or nonorientable complement. Indeed, if f is a power of a Dehn twist about a nonseparating simple closed curve c with nonorientable or orientable complement, then S^c is a nonorientable surface of genus $g - 2 \geq 3$ with holes and two boundary components or S^c is an orientable surface of genus $\frac{g-2}{2} \geq 1$ with holes and two boundary components, respectively. Since $g - 2 \geq 3$ or $\frac{g-2}{2} \geq 1$, there are nontrivial simple closed curves d and e on S^c meeting transversally at one point. Finally, the same argument used in the above paragraph finishes the proof. □

As mentioned in the introduction, algebraic characterizations of Dehn twists about nonseparating simple closed curves curves in both orientable (Theorem 2.1. of [4]) and nonorientable surfaces (provided that the curve is not characteristic, Theorem 3.1. of [1]; see also [2];) are already done. On the other hand, the situation for separating curves is more involved. First we remark that Theorem 3.2 and Theorem 3.3 of [1] are not correct as they are stated. E. Irmak informed us about the following counter example constructed by L. Paris.

A Counter Example. Let c be a separating curve in a surface S so that one of the two components of S^c is a torus with one boundary component $\Sigma_{1,1}$ as in the Figure 1. Consider the mapping class $\tau = (t_a t_b)^3$, where a and b are the curves given in the same figure. Let K be an abelian subgroup as in the statement of Theorem 3.2 or Theorem 3.3 of [1] for the Dehn twist

t_c. Since τ commutes with both t_a and t_b, τ is in $C_{\mathrm{M}(S)}(K)$. However, $t_c = \tau^2$ and hence t_c is not a primitive element in $C_{\mathrm{M}(S)}(K)$. Note that this example exists since the center of the mapping class group of $\Sigma_{1,1}$ is not trivial. Indeed it is isomorphic to the infinite cyclic group generated by τ. There are two more cases that might cause similar problem. The first one is $\Sigma_{1,1}^1$, whose mapping class group is isomorphic to that of $\Sigma_{1,1}$. The final problematic case is the surface $N_{1,1}^1$ whose mapping class group is the infinite cyclic group generated by the class of v, the class of the puncture slide diffeomorphism (Figure 1). In this case, v^2 is the Dehn twist about the boundary curve c.

Remark 2.3. The example we have just described above, also provides a counter example for Theorem 2.2 of [4], a result for algebraic characterization for Dehn twists (mainly, about separating curves) in orientable surfaces. The methods we will provide here, which work for both orientable and nonorientable surfaces, not only provide an algebraic characterization for Dehn twists about separating curves but also for the topological type of the separating curve the Dehn twist is about (see Section 5).

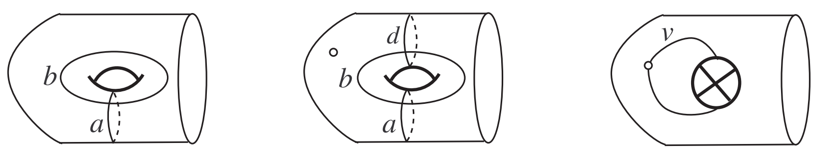

FIGURE 1.

3. PREPARATION FOR THE ALGEBRAIC CHARACTERIZATION OF DEHN TWIST ABOUT SEPARATING CURVES

We start with the following technical result which we will use later.

Proposition 3.1. *Let $S = N_g^k$ be a nonorientable surface of even genus. Then for any integer $s = 0, \cdots, \frac{g-2}{2}$, the group $\mathrm{M}(S)$ has an abelian subgroup of rank $\dfrac{3g-6-2s}{2} + k$, which is freely generated by Dehn twists about pairwise nonisotopic nonseparating simple closed curves, and so that no abelian subgroup containing this subgroup has bigger rank. Moreover, when we cut the surface along these curves, the resulting surface is a disjoint union of $g + k - 2s - 2$ many pair of pants and $2s$ many two holed real projective planes.*

Proof. Let $r = \dfrac{3g-6-2s}{2} + k$. By the maximality condition of the proposition, all the components of the surface cut along these r simple closed curves have Euler characteristic -1. In other words, each component is

either a pair of pants or a two holed real projective plane. Assume that there are l many pair of pants and m many two holed real projective planes. Hence, considering Euler characteristics we obtain the equation

$$2 - g - k = \chi(S) = \chi(\coprod_l N_0^3 \cup \coprod_m N_1^2) = -l - m \ .$$

On the other hand, counting the number of punctures, we obtain

$$3g - 6 - 2s + 3k = 3l + 2m \ .$$

These two equations yield $m = 2s$ and $l = g + k - 2s - 2$ as desired. Finally, the existence of such subgroups is readily seen by inspection. $\square$

3.1. **Separating chains and pairs of Dehn twists.** A sequence of Dehn twists $t_{a_1}, \cdots, t_{a_n}$, is called a chain if the following geometric intersection $i(a_i, a_{i+1}) = 1$, for $i = 1, \cdots, n-1$. The integer $n \geq 1$ is called the length of the chain. Note that if a chain has length more than one then each a_i must be two-sided nonseparating and has nonorientable complement. For a tree or chain of Dehn twists we always fix an orientation for a tubular neighborhood of the union of simple closed curves (which is always orientable) and consider Dehn twists using this orientation. It is known that two Dehn twists t_a, t_b satisfy the braid relation $t_a t_b t_a = t_b t_a t_b$ if and only if $i(a, b) = 1$ on nonorientable surfaces (see [9]). Hence, by the above results any automorphism $\Psi : \mathrm{M}(S) \to \mathrm{M}(S)$ maps a chain of Dehn twists of length at least two to another chain of Dehn twists of the same length. In this note, unless we state otherwise a chain or a tree in a nonorientable surface S will mean a chain or a tree of Dehn twists about nonseparating two-sided simple closed curves with nonorientable complement.

Below we will give a generalization of Lemma 3.7 of [1] to punctured surfaces. We will include the proof since the one in [1] has a gap, indicated by B. Szepietowski.

Lemma 3.2. *Let $S = N_g^k$ be a nonorientable surface of genus $g \geq 5$ with k. Then the image of a disc separating chain under an automorphism of* $\mathrm{M}(S)$ *is again a chain which separates a disc.*

Proof. If the genus of the surface is odd then a chain is separating if and only if it is maximal in $\mathrm{M}(S)$. However, being maximal is clearly preserved under an automorphism. Now Lemma 3.5 of [1] finishes the proof.

Now assume that the genus $g \geq 6$ is an even integer. Further assume that $c_1, \cdots, c_{2l+1}$ is a disc separating chain in $\mathrm{M}(S)$. Hence, when we delete a tubular neighborhood of the chain from the surface we obtain a disjoint union of a disc and a nonorientable surface, call S_0, of genus $g - 2l$ with k punctures and one boundary component. By Euler characteristic calculation and inspection we see that the group $\mathrm{M}(S_0)$ has a abelian subgroup K of rank $\dfrac{3(g-2l)-6}{2} + k + 2$, contained in each $C_{\mathrm{M}(S)}(t_{c_i})$, which is freely generated by Dehn twists about pairwise nonisotopic simple closed curves.

Now suppose on the contrary that the image $d_1, \cdots, d_{2l+1}$ of the chain $c_1, \cdots, c_{2l+1}$ under an automorphism is not separating. So by Lemma 3.5 of [1] the complement of a tubular neighborhood of the chain $d_1, \cdots, d_{2l+1}$ in S is an orientable surface of genus $\dfrac{g-2l}{2} - 1$ with k punctures and two boundary components, say c_1 and c_2. Let us call this surface S_1. (Since the surface S is nonorientable the curves c_1 and c_2 are both characteristic in the surface S.) The image of the abelian subgroup K under the same automorphism is again a maximal subgroup in $\mathrm{M}(S)$ and it lies in each $C_{\mathrm{M}(S)}(t_{d_i})$, $i = 1, \cdots, 2l+1$. However, the orientable subsurface S_1 can support an abelian subgroup K_0 in $\mathrm{M}(S)$, which lies in each $C_{\mathrm{M}(S)}(t_{d_i})$, of rank at most $\mathrm{rank}(K) - 1$. Some of the generators of both groups are Dehn twists about characteristic or separating curves. However, by Lemma 2.2 some powers of these generators are preserved under automorphisms. This finishes the proof. □

Now we will characterize algebraically a separating pair of Dehn twists about some two-sided simple closed curves, each of which is nonseparating with nonorientable complement (so that together they separate the surface).

Lemma 3.3. *Let $S = N_g^k$ be a nonorientable surface of genus $g \geq 5$ and a_1 and a_2 be disjoint, nonisotopic, nonseparating two-sided simple closed curves with nonorientable complements. Then a_1 and a_2 together separate the surface if and only if the following conditions are satisfied:*

(1) *For any Dehn twist t_b satisfying the braid relation $t_{a_1} t_b t_{a_1} = t_b t_{a_1} t_b$ we have $t_b \notin C_{\mathrm{M}(S)}(t_{a_2})$;*
(2) *In the even genus case, the twists t_{a_1} and t_{a_2} are contained in a free generating set, whose elements are all two-sided nonseparating simple closed curves with nonorientable complement, of a maximal abelian subgroup K in $\mathrm{M}(S)$, of rank $r = \frac{3g-6-2s}{2} + k$, where $s = 1$ or $s = 2$.*

Moreover, if the conditions of lemma are satisfied, then $s = 2$ if both components of the surface S cut along the curves a_1 and a_2 are nonorientable of even genus, and $s = 1$ otherwise.

Proof. Suppose first that a_1 and a_2 separate the surface. Now if a Dehn twist t_b satisfies the braid relation $t_{a_1} t_b t_{a_1} = t_b t_{a_1} t_b$ then the curves b and a_1 intersects geometrically once. However, since a_1 and a_2 separate the surface b and a_2 must intersect nontrivially and thus t_b cannot be contained in $C_{\mathrm{M}(S)}(t_{a_2})$. Moreover, it is easy to construct the required free abelian subgroup K, in case the genus of S is even.

For the other direction assume that the conditions of the lemma are satisfied, but on the contrary suppose that the surface cut along the curves a_1 and a_2 is connected. Hence, the surface S cut along only a_2, say S_0, is connected.

First we will treat the case where the genus g is an odd integer. Hence, the surface S_0 is a connected nonorientable surface of genus at least 3. Moreover, the curve a_1 is still nonseparating in S_0. Hence, there is a two-sided curve b in S_0, whose geometric intersection with a_1 is one. This is a contradiction to the first condition of the assumption.

Now let us consider the even genus case. By Proposition 3.1 the surface cut along the r many curves, about which the Dehn twists generate the subgroup K, has nonorientable components. Thus the surface S_0 is a connected nonorientable surface of genus at least four, and the curve a_1 is neither separating nor characteristic in S_0. Hence, as above there is a two-sided curve b in S_0, whose geometric intersection with a_1 is one. This finishes the proof for the even genus case.

The final part of the lemma is an immediate consequence of Proposition 3.1. □

3.2. **Triangles of Dehn twist.** Let a_1, a_2, a_3 be distinct, nonisotopic, nonseparating two-sided simple closed curves with nonorientable complements. We say that they form a triangle if each geometric intersection $i(a_i, a_j) = 1$, for all $i \neq j$ (see [2]). A triangle is called orientation reversing if there are Dehn twists about these curves, denoted by t_{a_1}, t_{a_2} and t_{a_3}, so that $t_{a_1}t_{a_2}t_{a_1} = t_{a_2}t_{a_1}t_{a_2}$, $t_{a_2}t_{a_3}t_{a_2} = t_{a_3}t_{a_2}t_{a_3}$ and $t_{a_1}^{-1}t_{a_3}t_{a_1}^{-1} = t_{a_3}t_{a_1}^{-1}t_{3_2}$. (Note that on a nonorientable surface we have exactly two Dehn twists about any two-sided circle a, which we may denote by t_a and t_a^{-1}.) Otherwise the triangle is called orientation preserving. Note that a triangle is either orientation preserving or orientation reversing but not both (cf. see Theorem 3.1 in [4]). Similarly, we will call the Dehn twists about these curves orientation reversing or preserving, respectively. It is clear that this property of triangles of curves or the Dehn twists about these curves is algebraic and thus it is preserved by automorphisms of the mapping class groups.

The following topological characterization is proved in [2].

Lemma 3.4. *The above triangle of Dehn twists is orientation preserving if and only if the union of these curves has an orientable regular neighborhood. Moreover, a regular neighborhood of a nonorientable triangle is $N_{4,1}$, a genus four nonorientable surface with one boundary component.*

3.3. **Dehn twists about characteristics curves and separating curves.** Let t_a be a Dehn twist about a separating simple closed curve a in a nonorientable surface S. Maximal chains contained in the centralizer $C_{\mathrm{M}(S)}(t_a)$ correspond to maximal chains of Dehn twists about two-sided simple closed curves (nonseparating with nonorientable complement) lying in different components of S^a. The lengths of these maximal chains determine the topological type the curve a up to a great extend, however fail to characterize its topological type completely. A more powerful tool is to use maximal trees contained in the centralizers. By a maximal tree of Dehn twists in a surface nonorientable S we will mean a connected maximal tree

of Dehn twists about pairwise nonisotopic nonseparating two-sided simple closed curves with nonorientable complements. If S is orientable we will only require that the curves in the tree to be nonseparating.

Remark 3.5. As it is seen in the figure below a maximal chain in a maximal tree need not to be a maximal chain in the surface. Note that the chain of circles $1, 2, \cdots, 7$ is maximal in the tree but not in the surface, which contains the longer chain $1, 2, \cdots, 7, 8, 9$.

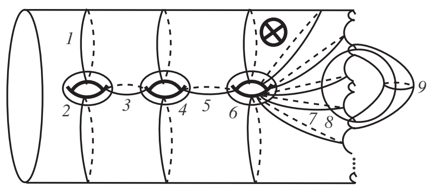

FIGURE 2.

The trees below will be useful for the rest of the paper: $T^k_{2g+1,1}$, $T^k_{2g+2,1}$ and $OT^k_{g,1}$.

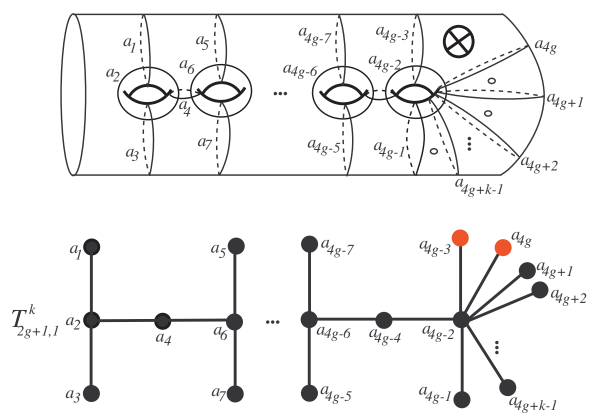

FIGURE 3.

We may endow an embedding of one of these trees into the group $\mathrm{M}(S)$, where S is a nonorientable surface of genus at least five, with a coloring of its vertices. For example, the tree $T^k_{2g+1,1}$ in Figure 3 has the vertices a_{4g-3} and a_{4g} colored. This will mean that any orientation reversing triangle in $\mathrm{M}(S)$, whose vertices commute with the colored vertices also commute with all the vertices of the tree. Equivalently, any orientation reversing triangle which lies in

$$C_{\mathrm{M}(S)}(t_{a_{4g-3}}) \cap C_{\mathrm{M}(S)}(t_{a_{4g}}) \ ,$$

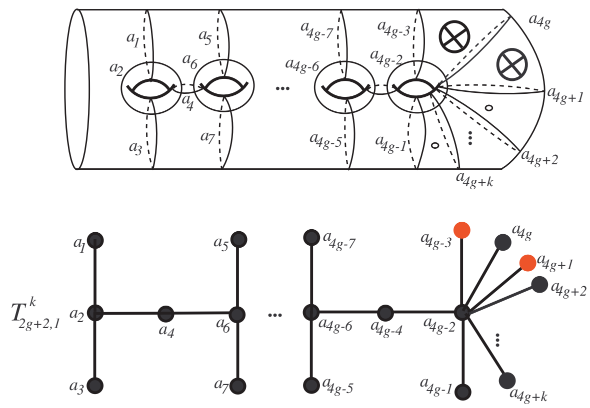

FIGURE 4.

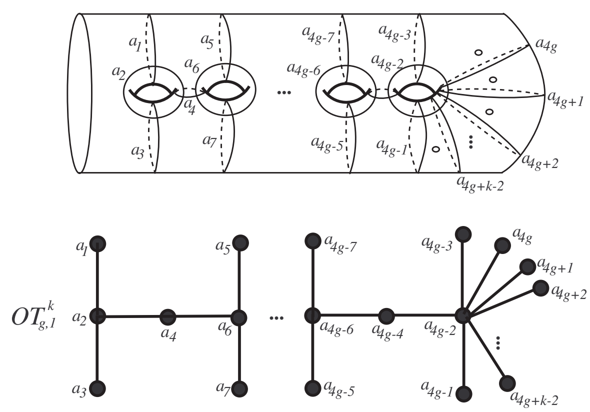

FIGURE 5.

also lies in $C_{\mathrm{M}(S)}(t_a)$, for all vertices a of the tree.

For nonorientable surfaces of even genera, first we present an algebraic characterization of a Dehn twist about a two-sided simple closed curve, whose complement is orientable.

Lemma 3.6. *Let $g \geq 2$, $k \geq 0$ be integers and c be a nontrivial two-sided simple closed curve in $S = N^k_{2g+2}$. Then S^c is orientable if and only if the colored tree $T = NT^k_{2g+2}$ (see Figure 6) can be embedded in the centralizer $C_{\mathrm{M}(S)}(t_c)$ as a maximal tree, where*

1. *each maximal chain in the tree is a maximal chain in $\mathrm{M}(S)$;*
2. *the tree T has a chain with length larger than or equal to any chain in the centralizer $C_{\mathrm{M}(S)}(t_c)$;*
3. *any two vertices connected to a_{4g-3}, both different than a_{4g-4}, form a separating pair.*

Proof. One direction is clear. Now assume that NT^k_{2g+2} lies in the centralizer $C_{\mathrm{M}(S)}(t_c)$. We claim that the surface S cut along the curves a_1, a_2 and a_3 has two components one of which is an orientable surface of genus $g-1$ with k punctures and one boundary component: To see this, consider a tubular neighborhood of the tree NT^k_{2g+2} with the vertex a_0 deleted. This is an orientable surface of genus g with $1+k+2(g-1)$ boundary components. By the maximality of the tree, $2g-2$ many of these components must bound discs. To illustrate this, consider, for example, the maximal chain a_1, a_2, a_4, a_6, a_5. The boundary component corresponding to this chain should bound a disc, because otherwise the tree would not be maximal (we could attach another two-sided simple closed curve to a_2). Note also that by the condition (3) of the hypothesis of the lemma the k pairs $(t_{a_{4g-2}}, t_{a_{4g-1}})$, $(t_{a_{4g-1}}, t_{a_{4g}})$, $\cdots$, $(t_{a_{4g+k-3}}, t_{a_{4g+k-2}})$ on the right corner of the tree are all separating. By maximality and the coloring they must all bound punctured annuli. This finishes the proof of the claim.

Now, if we attach a tubular neighborhood of the circle a_0 to this subsurface we obtain another subsurface, call S_0, of genus g with k punctures and two boundary components. Note that the two boundary components of S_0 should be glued so that the resulting surface would be nonorientable of genus $2g+2$ with k punctures. Finally, c being disjoint from each vertex of the tree implies that c lies in $S - \cup_{i\geq 1} a_i$, which is a disjoint union of $2g$ discs, k punctured annuli and a one holed Klein bottle. In particular, up to isotopy, c is the unique nontrivial two-sided simple closed curve in this holed Klein bottle. This finishes the proof.

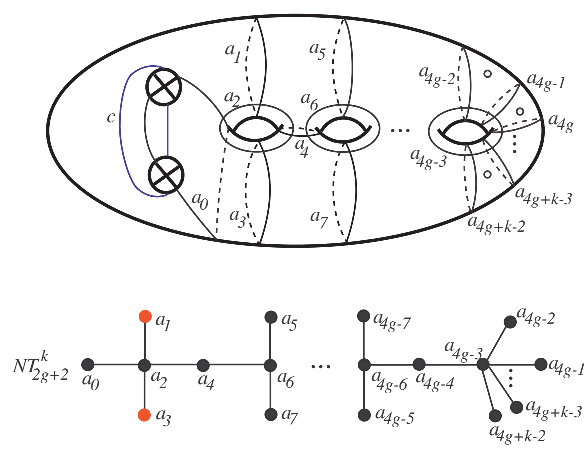

FIGURE 6.

□

We note that the above lemma and Lemma 2.2 yield the following algebraic characterization for powers of Dehn twists about nonseparating simple

closed curves with orientable complement on nonorientable surfaces of even genus.

Corollary 3.7. *Let $g \geq 2$, $k \geq 0$ be integers f a mapping class in $\mathrm{M}(S)$ such that $f = t_c^m$ for some integer $m > 0$, and a nontrivial simple closed curve c on $S = N_{2g+2}^k$. Then c is a characteristic curve (i.e., its complement S^c is orientable) if and only if the colored tree $T = NT_{2g+2}^k$ (see Figure 6) can be embedded in the centralizer $C_{\mathrm{M}(S)}(t_c)$ as a maximal tree, where*

1. *each maximal chain in the tree is a maximal chain in $\mathrm{M}(S)$;*
2. *the tree T has a chain with length larger than or equal to any chain in the centralizer $C_{\mathrm{M}(S)}(t_c)$;*
3. *any two vertices connected to a_{4g-3}, both different than a_{4g-4}, form a separating pair.*

For separating Dehn twists characterization we will make use of the following lemma.

Lemma 3.8. *Let g be a positive integer and T be one of the colored trees $T_{2g+1,1}^k$, $T_{2g+2,1}^k$ or $OT_{g,1}^k$, embedded in the group $\mathrm{M}(S)$, where S is a nonorientable surface of genus at least five. Suppose that c is a nontrivial separating simple closed curve in S and the tree T lies in the centralizer $C_{\mathrm{M}(S)}(t_c)$ as a maximal tree. Moreover, assume the followings:*

(1) *Each maximal chain in the tree is a maximal chain in $\mathrm{M}(S)$;*
(2) *If the tree T is $T_{2g+1,1}^k$ or $T_{2g+2,1}^k$, then it has a chain with length larger than or equal to any chain in the centralizer $C_{\mathrm{M}(S)}(t_c)$.*
(3) *Any two vertices connected to a_{4g-2}, except a_{4g-4}, form a separating pair.*
(4a) *If $T = T_{2g+1,1}^k$ then $g \geq 2$, and if $T = T_{2g+2,1}^k$ then $g \geq 1$. Moreover, in both cases, there is an orientation reversing triangle in $C_{\mathrm{M}(S)}(t_c)$, which is not contained in*

$$\bigcap_{a_i \in V(T)} C_{\mathrm{M}(S)}(t_{a_i}),$$

where $V(T)$ is the set of vertices of T;
(4b) *If $T = OT_{g,1}^k$ then $g \geq 2$, and any orientation reversing triangle in $\mathrm{M}(S)$ also lies in*

$$C_{\mathrm{M}(S)}(t_{a_{4g-3}}) \cap C_{\mathrm{M}(S)}(t_{a_{4g-1}}) \ .$$

Then S^c has a component homeomorphic to $N_{2g+1,1}^k$, $N_{2g+2,1}^k$ or $\Sigma_{g,1}^k$, respectively.

Proof. Case 1: $T = T_{2g+1,1}^k$. First we assume that S is of odd genus. For each Dehn twist belonging the tree choose a two-sided nonseparating simple closed curve a_i with nonorientable complement. Let S_0 be the closure of a tubular neighborhood of the tree of the curves a_i. Then S_0 is an orientable subsurface of S with Euler characteristic $\chi(S_0) = 2-4g-k$ and with $2g+k$

boundary components. So S_0 is an orientable surface of genus g with $2g+k$ boundary components. By the definition of maximal tree each maximal chain in T is a maximal chain in the surface S. Moreover, S has odd genus and thus each maximal chain contained in T separates the surface (this is the only place we use the assumption that S is of odd genus). Hence, each boundary component bounds either a disc, a once punctured disc, an annulus or a Möbius band. Again by maximality of the tree, $2g-2$ many of these components must bound discs. To illustrate this, consider for example, the maximal chain a_1, a_2, a_4, a_6, a_5. The boundary component corresponding to this chain should bound a disc, because otherwise the tree would not be maximal (we could attach another two-sided simple closed curve to a_2).

Note that by the condition (3) of the hypothesis of the lemma the $k+1$ pairs $(t_{a_{4g-3}}, t_{a_{4g}})$, $(t_{a_{4g}}, t_{a_{4g+1}})$, $\cdots$, $(t_{a_{4g+k-1}}, t_{a_{4g-1}})$ on the right corner of the tree are all separating.

The condition (2) of the hypothesis of the lemma implies that at most two of these pairs may bound a projective plane with two boundary components. On the other hand, the condition (4a) implies that at least one of them bounds a projective plane with two boundary components. Finally, the coloring of the vertices implies that boundary components corresponding to the chain a_{4g-3}, a_{4g} is the only pair that bounds a projective plane with two boundary components. Hence, the other k separating pairs must bound punctured annuli.

By attaching $2g-2$ discs and k punctured discs and a Möbius band to S_0 we get a nonorientable surface, say S_1, of genus $2g+1$ with one boundary component. S_1 is contained as a subsurface in one of the two components of $S^c = S_2 \cup S_3$, say in S_2. Since S_2 has only one boundary component, the boundary component corresponding to the chain a_1, a_2, a_3 must be parallel to the boundary of S_2. Hence, we are done in the odd genus case.

Now let us consider the case where S has even genus. If each maximal chain in T is separating in S then the above proof works in this case as well. Now we will show that any maximal chain in T is indeed separating in S. To prove this, assume that there is a maximal chain $t_{c_1}, \cdots, t_{c_{2l+1}}$ in T, and thus in $\mathrm{M}(S)$, so that the chain of two-sided simple closed curves $c_1, \cdots, c_{2l+1}$ is not separating in S. A tubular neighborhood ν of the union of these curves is an orientable surface of genus l with two boundary components. Since the chain is maximal in $\mathrm{M}(S)$ we see that the surface $S \setminus int(\nu)$ is an annulus, possibly with punctures, so that when the orientable surfaces ν and $S \setminus int(\nu)$ are glued along the two boundary components, the resulting surface S is nonorientable (see Figure 7) with genus $2l+2$. Since the separating curve c is disjoint from the tree and thus from the chain, we see that c lies in the annulus and bounds a (at least twice) punctured disc.

So the topological type of c is determined up to the number of punctures of S contained in either sides of c. On the other hand, since the surface S has even genus, exactly two of the separating pairs $(t_{a_{4g-3}}, t_{a_{4g}})$, $(t_{a_{4g}}, t_{a_{4g+1}})$,

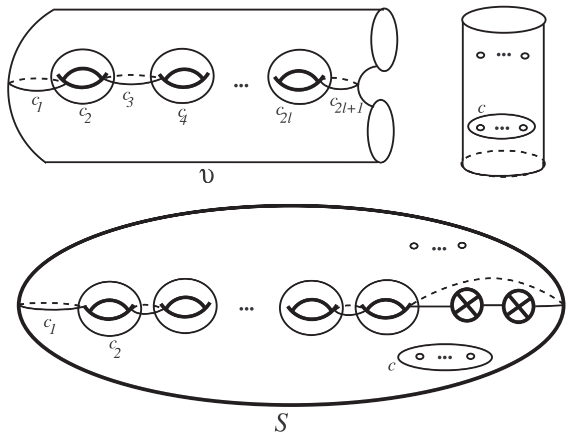

FIGURE 7.

$\cdots$, $(t_{a_{4g+k-1}}, t_{a_{4g-1}})$ bound a projective plane with two boundary components. On the other hand, the coloring of the vertices implies that a_{4g-3}, a_{4g} is the only pair bound a projective plane with two boundary components. This gives the desired contradiction. Hence the proof finishes in the case where $T = T^k_{2g+1,1}$.

Case 2: $T = T^k_{2g+2,1}$. Now first assume that the surface S has odd genus. We proceed analogous to the previous case and arrive at the point, where the $k+2$ pairs $(t_{a_{4g-3}}, t_{a_{4g}})$, $(t_{a_{4g}}, t_{a_{4g+1}})$, $\cdots$, $(t_{a_{4g+k}}, t_{a_{4g-1}})$ on the right corner of the tree are all separating. Again the condition (2) of the hypothesis of the lemma implies that at most two of these pairs bound a projective plane with two boundary components and the others will bound punctured annuli. Then condition (4a) and the coloring of the vertices imply that exactly the two pairs $(t_{a_{4g-3}}, t_{a_{4g}})$ and $(t_{a_{4g}}, t_{a_{4g+1}})$ bound projective plane with two boundary components and all other pairs bound punctured annuli. This finishes the proof for the odd genus case.

The even genus case is again analogous to that in Case 1. We just need to show that any maximal chain in the tree is separating. We proceed as in case $T = T^k_{2g+1,1}$. Without loss of generality we may assume that c_{2l+1} belongs to the set

$$\{a_{4g-3}, a_{4g}, a_{4g+1}, \cdots, a_{4g+k}, a_{4g-1}\} .$$

The condition (2) of the hypothesis of the lemma implies that the $k+2$ pairs $(t_{a_{4g-3}}, t_{a_{4g}})$, $(t_{a_{4g}}, t_{a_{4g+1}})$, $\cdots$, $(t_{a_{4g+k}}, t_{a_{4g-1}})$ on the right corner of the tree are all bounding punctured annuli. However, this contradicts to the maximality of the tree, since in this case we may add two more vertices to the tree as in Figure 4. So we are done in this case too.

Case 3: $T = OT^k_{g,1}$. This case is easy now, because by condition (4b) all the separating pairs on the right side of the tree will bound punctured annuli (see the paragraph below Remark 3.5). It follows that the surface S_1, in this case, will be an orientable surface of genus g with one boundary component and with k punctures (compare with the surface S_1 we obtained in the case $T = T^k_{2g+1,1}$ above). $\square$

Remark 3.9. 1) One needs to be careful when using the above lemma to characterize the topological type of the curve (or the Dehn twist t_c) algebraically: Namely, if one of the components of S^c, the surface S cut along the curve c, is orientable of genus at least two then we can embed $T = OT^k_{g,1}$ into the centralizer $C_{\mathrm{M}(S)}(t_c)$ satisfying the conditions of the lemma so that the topological type of the curve (or the Dehn twist t_c) is algebraically characterized. If the orientable component is of genus one, then the intersection of $C_{\mathrm{M}(S)}(t_c)$ with the centralizer of the tree (the intersection of all the centralizers of the vertices of T) contains a pair of Dehn twists t_a and t_b, about nonseparating curves with nonorientable complements so that t_a and t_b satisfy the braid relation.

On the other hand, if both components of S^c are nonorientable, then we have to choose the component of S^c so that condition (2) of the lemma is satisfied, which is always possible if $g \geq 7$ (see also the next theorem). In other words, we need to choose the component which contains the longer chain.

2) For odd genus surfaces S the condition (3) of the hypothesis of the lemma is void. For example consider the pair $(t_{a_{4g-3}}, t_{a_{4g}})$ in the first tree $T^k_{2g+1,1}$. The maximal chain $t_{a_{4g-3}}, t_{a_{4g-2}}, t_{a_{4g}}$ in T is maximal in the surface and thus is separating. Hence, a tubular neighborhood of the chain $t_{a_{4g-3}}, t_{a_{4g-2}}, t_{a_{4g}}$ is a torus with two boundary components, each of which is a separating curve. This implies that the pair $(t_{a_{4g-3}}, t_{a_{4g}})$ is separating.

4. Completing the proof of the algebraic characterization of Dehn twist about separating curves

We note that Dehn twists about two-sided nonseparating simple closed curves with nonorientable complements is already known ([2] or [1]). (See Theorem 2.1 in [4] for characterization for Dehn twists about nonseparating simple closed curves on orientable surfaces.) Moreover, by Lemma 2.2 and Corollary 3.7 it is enough to characterize Dehn twist about separating curves algebraically, assuming already that the element have the form $f = t_c^m$, for some integer $m > 0$ and a nontrivial separating simple closed curve c on $S = N^k_g$. In this case, one can easily see that there exists a free abelian subgroup K of $\mathrm{M}(S)$ generated by f and $\dfrac{3g-9}{2} + k$, when g is odd, (respectively, $\dfrac{3g-10}{2} + k$, when g is even), twists about two-sided nonseparating simple closed curves with nonorientable complements such

that $rank\ (K) = \dfrac{3g-7}{2} + k$, when g is odd, (respectively, $\dfrac{3g-8}{2} + k$, when g is even).

Also note that if genus $g \geq 7$ then one of the two components of S^c is either a nonorientable surface of genus at least genus four or an orientable surface of genus at least two. Hence, the above lemma and the remark following it are applicable.

Theorem 4.1. *Let $g \geq 5$, $k \geq 0$ be integers f a mapping class in* $\mathrm{M}(S)$ *such that $f = t_c^m$ and K be as above. If Lemma 3.8 is applicable, which is always the case if $g \geq 7$, then $f = t_c$ if and only if*

(1) *f is a primitive element of $C_{\mathrm{M}(S)}(K)$ if c does not separate $\Sigma_{1,1}$ $\Sigma_{1,1}^1$ or $N_{1,1}^1$;*
(2) *$f = (t_a t_b)^6$ if c separates $\Sigma_{1,1}$, where t_a, t_b is a chain contained in the intersection of $C_{\mathrm{M}(S)}(t_c)$ with the centralizer of the tree which is used in Lemma 3.8 (see Figure 1);*
(3) *$f = (t_a t_b t_d)^4$ if c separates $\Sigma_{1,1}^1$, where t_a, t_b, t_d is a chain of Dehn twists, about nonseparating simple closed curves with nonorientable complements, where all are contained in the intersection of $C_{\mathrm{M}(S)}(t_c)$ with the centralizer of the tree;*
(4) *$f = v^2$ if c separates $N_{1,1}^1$, where v is the class of the puncture slide diffeomorphism, a generator of the intersection of $C_{\mathrm{M}(S)}(t_c)$ with the centralizer of the tree.*

Moreover, the topological type of c is determined completely via Lemma 3.8.

If Lemma 3.8 *is not applicable and $g = 6$ then $f = t_c$ if and only if f is primitive in $C_{\mathrm{M}(S)}(K)$. Furthermore, each component of S^c is a nonorientable surface of genus three and the number punctures in each component is $r_i - 2$, where r_i, $i = 1, 2$ are determined by the maximal trees in Figure 8 below can be embedded in $C_{\mathrm{M}(S)}(t_c)$.*

Finally, if Lemma 3.8 *is not applicable and $g = 5$ then one of the components is again a nonorientable surface of genus three. The topological type of the other component is also algebraically characterized.*

Proof. First assume that Lemma 3.8 is applicable. If c does not separate $\Sigma_{1,1}$ $\Sigma_{1,1}^1$ or $N_{1,1}^1$ then we are done because the arguments provided in the proof of Theorem 2.2 of [4] will work in this case since the problematic cases are ruled out.

If c separates $\Sigma_{1,1}$ then f is a Dehn twist (i.e., $f = t_c$) if and only if $f = (t_a t_b)^6$ for some Dehn twists t_a and t_b, about nonseparating simple closed curves with nonorientable complements so that t_a, t_b is a chain (i.e., they satisfy the braid relation), where both are contained in the intersection of $C_{\mathrm{M}(S)}(t_c)$ with the centralizer of the tree which is used in Lemma 3.8 (see Figure 1).

Similarly, if c separates $\Sigma_{1,1}^1$ then f is a Dehn twist if and only if $f = (t_a t_b t_d)^4$ for a chain of Dehn twists t_a, t_b, t_d, about nonseparating simple

closed curves with nonorientable complements, where all are contained in the intersection of $C_{\mathrm{M}(S)}(t_c)$ with the centralizer of the tree (see Figure 1).

Finally, if c separates $N_{1,1}^1$ then f is a Dehn twist if and only if $f = v^{\pm 2}$, where v is the class of the puncture slide diffeomorphism, again contained in the intersection of $C_{\mathrm{M}(S)}(t_c)$ with the centralizer of the tree (we know that $t_c = v^2$; see Figure 1). This provides an algebraic characterization because v is a generator for the mapping class group of $N_{1,1}^1$, which is isomorphic to the intersection of $C_{\mathrm{M}(S)}(t_c)$ with the centralizer of the tree.

If $g = 5$ or 6 and the hypothesis of Lemma 3.8 is still satisfied then again there is nothing to do. Hence, we are left with the cases $g = 5$ or 6 and Lemma 3.8 is not applicable. Hence, one of the components of S^c is nonorientable of genus three.

In case $g = 6$, first note that, both components of S^c are nonorientable of genus three and thus $f = t_c$ if and only if f is primitive in $C_{\mathrm{M}(S)}(K)$. Moreover, via Euler characteristic considerations, the maximal trees in Figure 8 contained in $C_{\mathrm{M}(S)}(t_c)$ determines the number of punctures in each component. Indeed, it is $r - 2$ if r is as in the theorem.

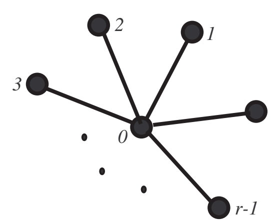

FIGURE 8.

If $g = 5$ we proceed as follows: We know that one component is a punctured nonorientable surface of genus three. Thus we need a way of checking algebraically whether the second component is a puncured torus or a Klein bottle. A punctured nonorientable surface of genus three supports a maximal chain of length three. A (punctured) torus supports a chain of length at least two and a punctured Klein bottle supports no chain of length two. Hence, if $C_{\mathrm{M}(S)}(t_c)$ contains two maximal chains of lengths at least two, where any Dehn twist from one chain commutes with any of the Dehn twists of the other chain, then the second component of S^c is a punctured torus. Otherwise the other component is a punctured Klein bottle. The number of punctures in each component can be determined as in the genus six case. Note that the Dehn twists in Figure 8 contained in one component should commute with all the Dehn twists with the chain in the other component. This way we can determine the number of punctures in each component.

If the second component is a punctured Klein bottle or a punctured torus with more than one punctures then $f = t_c$ if and only if f is primitive in

$C_{\mathrm{M}(S)}(K)$. If the second component is torus with at most one puncture then we proceed as in the parts (2) and (3) above. □

5. Orientable Surfaces

In this section, analogously to the above section, we will give an algebraic characterization for Dehn twists about separating curves on an orientable surface of genus at least three. We will omit the proofs since they are essentially the same. In fact, the versions for orientable surfaces will be even easier since a large portion of our efforts has been spent to distinguish algebraically an annulus with one puncture from the projective plane with two boundary components. First let us give a version of Lemma 3.8 for orientable surfaces.

Lemma 5.1. *Let g be a positive integer and T be the colored tree $OT^k_{g,1}$, embedded in the group $\mathrm{M}(S)$, where S is an orientable surface of genus at least two. Suppose that c is a nontrivial separating simple closed curve in S and the tree T lies in the centralizer $C_{\mathrm{M}(S)}(t_c)$ as a maximal tree. Moreover, assume the followings:*

1. *Each maximal chain in the tree is a maximal chain in $\mathrm{M}(S)$;*
2. *Any two vertices connected to a_{4g-2}, except a_{4g-4}, form a separating pair.*

Then S^c has a component homeomorphic to $\Sigma^k_{g,1}$.

Now we can state the characterization result for separating Dehn twists:

Theorem 5.2. *Let $g \geq 3,\ k \geq 0$ be integers f a mapping class in $\mathrm{M}(S)$ such that $f = t_c^m$ for some integer $m > 0$ and a nontrivial separating simple closed curve c on the orientable surface $S = \Sigma^k_g$. If Lemma 5.1 is applicable, which is always the case if $g \geq 4$, then $f = t_c$ if and only if*

1. *f is a primitive element of $C_{\mathrm{M}(S)}(K)$ if c does not separate $\Sigma_{1,1}$ or $\Sigma^1_{1,1}$;*
2. *$f = (t_a t_b)^6$ if c separates $\Sigma_{1,1}$, where t_a, t_b is a chain contained in the intersection of $C_{\mathrm{M}(S)}(t_c)$ with the centralizer of the tree which is used in Lemma 5.1 (see Figure 1);*
3. *$f = (t_a t_b t_d)^4$ if c separates $\Sigma^1_{1,1}$, where t_a, t_b, t_d is a chain of Dehn twists, about nonseparating simple closed curves, where all are contained in the intersection of $C_{\mathrm{M}(S)}(t_c)$ with the centralizer of the tree (see Figure 1).*

Moreover, the topological type of c is determined completely via Lemma 5.1.

If Lemma 5.1 *is not applicable then the curve separates a punctured torus. The number punctures in each component can be determined the same way as in Theorem 4.1. If the number of punctures is at least two then $f = t_c$ if and only if f is primitive. Otherwise, parts (2) and (3) of this theorem applies.*

6. Some Applications

Let $S = N_g^k$. The subgroup $\mathcal{T}$ of the mapping class group generated by all Dehn twists about two-sided simple closed curves, is called the twist subgroup. It is known that this subgroup is of index $2^{k+1}k!$ in $\mathrm{Mod}(S)$, provided that $g \geq 3$ ([7], [10]). Now, as a consequence of characterization results of Dehn twists we state the following corollary:

Corollary 6.1. *For $g \geq 5$, let $\Phi : \mathrm{M}(S) \to \mathrm{M}(S)$ be an automorphism and $\mathcal{T} \leq \mathrm{M}(S)$ be the twist subgroup. If $t_c \in \mathcal{T}$ is a Dehn twist then so is $\Phi(t_c)$. Moreover, the Dehn twists t_c and $\Phi(t_c)$ are topologically equivalent. In other words, there is a homeomorphism $f : S \to S$ such that $\Phi(t_c) = t_{f(c)}$. In particular, the twist subgroup is a characteristic subgroup of* $\mathrm{M}(S)$.

The subgroup $\mathrm{PMod}^+(S)$ has index 2^k in $\mathrm{PMod}(S)$ and contains the twist subgroup $\mathcal{T}$ as a subgroup of index two ([10]) provided that $g \geq 3$.

Lemma 6.2. *The subgroup* $\mathrm{PMod}^+(S)$ *is characteristic in* $\mathrm{Mod}(S)$ *and* $\mathrm{PMod}(S)$, *provided that $g \geq 5$.*

Proof. We know that the twist subgroup is characteristic in all the three groups in the statement of the lemma. On the other hand, $\mathrm{PMod}^+(S)$ is generated by the twist subgroup and the set of all Y-homeomorphisms, each of which is supported inside Klein bottles with one boundary component, so that both components of the surface cut along this boundary curve are nonorientable. It is easy to see that this set is also characteristic (cf. Theorem 3.9 and Theorem 3.10 of [1]). Indeed, one can see this directly as follows: Note that, in the Klein bottle with the boundary circle e, a Y-homeomorphism represents a mapping class, say τ, which is characterized as a mapping class that is not a Dehn twist but $\tau^2 = t_e$. It follows that $\mathrm{PMod}^+(S)$ is characteristic in $\mathrm{Mod}(S)$ and $\mathrm{PMod}(S)$. □

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Department of Mathematics, Atilim University, 06836 Ankara, TURKEY
E-mail address: ferihe.atalan@atilim.edu.tr